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COMPUTATION OF TRANSIENT TEMPERATURES IN REGENERATORS

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Abstract—A non-iterative method for computing the temperature variations in a regenerator at cyclic equilibrium has been developed by earlier workers. Previous attempts to apply this technique have produced inaccurate results, but the present authors have obtained reasonable results for selected examples without diminishing the step size for numerical integration. This has been achieved by ensuring that certain infinite series arising in the computation are summed accurately, and by selecting sufficiently accurate forms for numerical integration.

The method was previously derived for regenerators operating with equal heating and cooling periods, but has now been modified for the more general case. The direction of further development is indicated.

NOMENCLATURE*

a, semi-thickness of regenerator wall;

a', separation of regenerator walls;

a_1, $\dfrac{N_1}{N_1 - 1}$;

a_2, $\dfrac{N_2}{N_2 - 1}$;

a_3, $\dfrac{-1}{N_1 - 1} + \dfrac{1}{N_2 - 1}$;

a_4, $\dfrac{1}{N_1 - 1} + \dfrac{1}{N_2 - 1}$;

A, constant introduced in equation (45);

A_1, A_2, matrices arising from approximate forms of definite integrals [see equations (29) and (30)];

A'_1, A'_2, matrices defined in equations (35) and (36);

b_s, $\dfrac{2\beta_s}{2 + \beta_s}$;

* Some symbols are used differently in the Appendix. See note at beginning of Appendix.

B, constant introduced in equation (45);

B_1, B_2, matrices arising from numerical approximation of definite integrals and introduced in equations (82) and (83);

C, $\tau - \Omega$, or τ depending on the value of z';

d_1, $\dfrac{1}{\Omega_1(N_1 - 1)}$;

d_2, $\dfrac{1}{\Omega_2(N_2 - 1)}$;

D_1, D_2, matrices arising from approximation of definite integrals and introduced in equations (82) and (83);

$f(\zeta, z')$, $f_1(\zeta, z')$, $f_2(\zeta, z')$ $\chi(\zeta, z)$, $\chi_1(\zeta, z)$, $\chi_2(\zeta, z)$, transformed to axes (ζ, z'); [In the section dealing with unequal heating and cooling periods χ_i, (for $i = 1$ or 2) transforms to $f_i(\zeta, v)$];

$\bar{f}_i$, mean value of f_i in the appropriate period, i;

$\overline{\delta f}_i$, time mean change in f_i across the whole regenerator (measured from cold end to hot end);

F_i ($i = 1$ or 2), matrices formed from values of f_i at discrete values of z';

$F_{i,r}$, F_i at $\zeta = (r/m)$;

F', column matrix $\{F_2, F_1\}$;

F'_r, F' at $\zeta = (r/m)$;

$F'^{(s)}_{-m}$, for $1 \leqslant s \leqslant n+2$, one of $(n+2)$ hypothetical forms of F'_{-m} [see equations (67)];

$F'^{(s)}_r$, the solution for F' at $\xi = (r/m)\,(-m \leqslant r \leqslant +m)$ which satisfies the hypothetical boundary state $F'^{(s)}_{-m}$;

$F'^{(s)}_{r,j}$, the jth element of the column matrix $F'^{(s)}_r$;

$g(v)$, function defined in equation (81);

h, (Ω/n), time interval for numerical integration;

h_i, coefficient of heat transfer between fluid and solid in appropriate period i;

h_p, overall heat transfer coefficient per cycle;

H_j, for $1 \leqslant j \leqslant 6$, functions defined in equations (54)–(56) and (58)–(60);

i, subscript to distinguish heating period from cooling period; $i = 1$ during the heating period and $i = 2$ during the cooling period;

I, unit matrix;

I_i, definite integrals defined in equations (37) and (41);

δI_i, contribution to I_i from an element of duration h;

j, an integer;

k, thermal conductivity of solid filling in the direction perpendicular to fluid flow;

K_s, for $1 \leqslant s \leqslant n+2$, constant coefficient in equation (68);

$\bar{l}$, width of solid wall in real regenerator;

L, length of regenerator channel;

$2m$, integral number of subdivisions of regenerator length;

M_i, $(2W_iS_ia/kpL)$, dimensionless parameter, characteristic of appropriate period i;

n, integral number of steps into which each period is subdivided for stepwise approximate integration;

$\bar{n}$, an integer;

N_i, (h_ia/k), Nusselt number characteristic of appropriate period i;

$\bar{N}$, number of walls in real regenerator;

p, channel perimeter;

P, duration of heating or cooling period when these are equal;

P_i, duration of appropriate period, i;

P', a matrix defined by equation (62);

P'_r, P' at $\zeta = (r/m)$;

Q_i, function defined in equation (95);

r, an integer. Also equations (21) and (22) are applied at discrete values of τ defined by $\tau = (\Omega r/n)$;

r', an integer;

R_i, function defined in equation (96);

s, an integer;

S_i, specific heat of fluid in appropriate period, i;

$t(y, \theta)$, current fluid temperature in either period;

$\bar{t}_i$, time-mean value of t in appropriate period, i;

$t'(y, \theta)$, as $t(y, \theta)$ but dimensionless;

$t'_i(y, \theta)$, $t'(y, \theta)$ in appropriate period i;

T, solid temperature;

T', as T, but dimensionless;

T_1, T_2, matrices originating in equation (32) and defined by equation (34);

U, used generally to signify t, t_1, t_2, or T;
U', magnitude of U in dimensionless units;
U_i, functions defined in equation (46);
$\overline{U}_i$, a function defined in equation (49);
v, dimensionless time defined in equation (72);
V_1, V_2, matrices defined in equations (89) and (90);
w_i, mass of fluid occupying unit length of regenerator channel;
W_i, mass flow rate of fluid per channel;
x, distance from solid surface into regenerator wall;
X_1, function defined by equation (23);
X_2, function defined by equation (24);
$X'_1, (X'_2)$, matrix constructed from values of $X_1, (X_2)$ at discrete values of ζ and z';
y, distance along regenerator channel, measured from mid-point;
z, $\frac{\alpha\theta}{a^2}$, dimensionless time scale;
z', z changed to an origin at the proceeding reversal;
Z_1, Z_2, matrices defined by equations (91) and (92);
α, diffusivity of regenerator wall material;
β_s, sth root of $\sqrt{\beta} \tan \sqrt{\beta} = 1$;
γ_1, γ_2, matrices occurring in equation (31) and defined by equations (33);
ζ, $(2y/L)$, dimensionless distance along regenerator channel, measured from mid-point;
η, η_1, η_2, functions defined by equations (25)–(27);
θ, time, measured from the commencement of a heating period;
ξ, (x/a), dimensionless measure of distance from solid surface;
$\Xi(\xi, \zeta, z)$, as T', but referred to dimensionless independent variables ξ, ζ and z;
ρ_{irs} $(i = 1$ or $2)$, multipliers defined in equations (38) and (42);
τ, τ_1, particular instantaneous values of z';
$\bar{\tau}$, (τ/Ω_1), dimensionless time scale;
ϕ_1, ϕ_2, functions defined by equations (79) and (80);
ϕ'_1, ϕ'_2, column matrices whose elements are formed from discrete values of ϕ_1 and ϕ_2;
$\chi(\zeta, z), \chi_1(\zeta, z)$ and $\chi_2(\zeta, z)$, $t'(y, \theta), t'_1(y, 0)$, and $t'_2(y, \theta)$ transformed to variables ζ and z;
Ω, used for Ω_1 or Ω_2 when these are equal;
Ω_1, $(\alpha_1 P_1/a^2)$, $\Omega_2 = (\alpha_2 P_2/a^2)$, dimensionless parameters characterising duration of hot and cold periods respectively (cf. z).

INTRODUCTION

A REGENERATOR is a device which transfers heat from a hot medium to a cooler medium by means of an intervening heat store. The particular arrangement treated here is illustrated in Fig. 1a, where a fluid flows over a solid matrix for a time P_1, after which a relatively cooler fluid flows in the reverse direction for time P_2. This process is then repeated for all time, the flow rates and entry temperatures of both fluids being periodic functions of time. The system eventually settles to a state where, from an arbitrary origin of time, fluid and solid temperatures are repeated exactly after each time interval of $(P_1 + P_2)$. This state is termed "cyclic equilibrium", or, when stated as a mathematical condition, the term "reversal condition" is used.

This problem has been treated by several

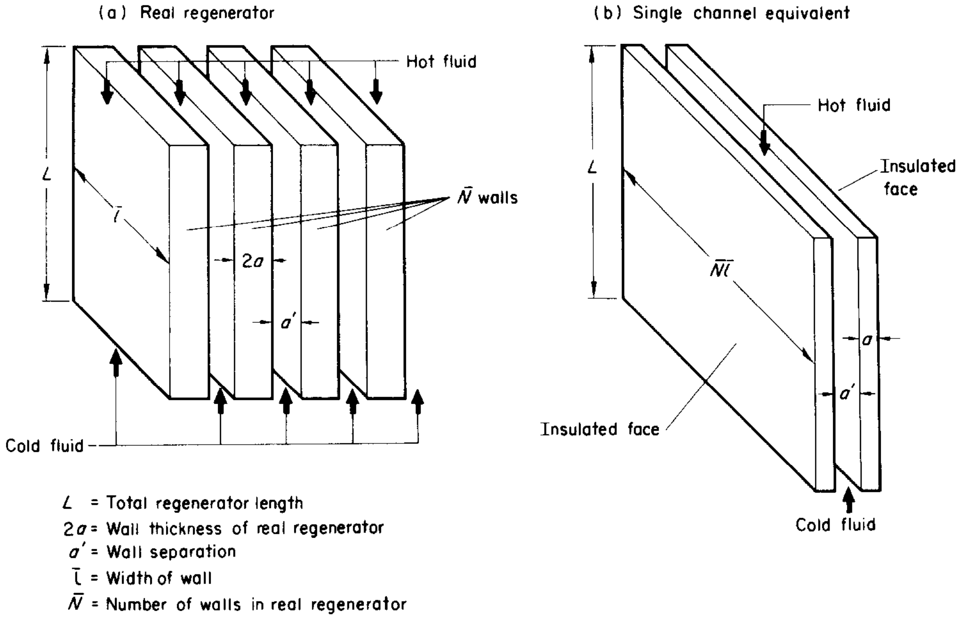

FIG. 1. Relation between real regenerator and equivalent single channel regenerator.

workers, the earliest being Nusselt [1], who obtained closed solutions for some simplified cases. Later workers have made simplifying mathematical approximations [2], or have used empirical values for the overall heat transfer coefficient, h_p [3], while this same empirical step is implicit in other solutions of more complex mathematical derivation [4, 5]. The most successful treatment of a purely mathematical nature is undoubtedly that due to Hausen [6], who extrapolated certain relationships from the perfectly conducting wall instance to the case for a wall with finite conductivity. Opinion is divided as to whether or not this extrapolation is exact. (In the present authors' view this has not been proved, and Hausen himself used the term "improved calculations", rather than "exact calculations".) Hausen's results were expressed in terms of two parameters, which were dimensionless representations of the reversal time and the surface area. The general case, where these parameters are unequal in the heating and cooling periods, was not analysed rigorously but Hausen suggested using geometric mean values under these conditions. Comparison with results obtained by digital simulation [7] and with results reported in this paper confirm that the method proposed by Hausen holds to a good degree of accuracy.

The various methods that have been proposed for calculating the thermal performance of regenerators have been surveyed recently [8]. The present research is concerned particularly with the development of a technique for digital computation of a regenerator. This approach will be useful when several computations are required, such as for an optimisation exercise. Also design detailing, such as the selection of materials may require a detailed knowledge of temperature distribution, rather than merely an overall performance factor. Even using a simplified approach for hand calculations, the process can be laborious [9], and so a technique using a digital computer may be justified even for single calculations, once a suitable program has been developed.

The method used by the authors is a development of that originally proposed by Collins and Daws [10]. The main advantage of this method over that of complete digital simulation is that the reversal condition is satisfied analytically, thus eliminating the need for simulating successive cycles. The computation time is, therefore, predictable, and there is no problem of confusion between slow convergence and cyclic equilibrium.

Application of the Collins and Daws' method has previously been limited to three computed examples. The first computation was carried out by Taylor [10] on a desk machine, and the results showed a thermal balance deficiency of 10 per cent. Of the two examples computed by Albasiny [11], using an electronic digital computer, one gave results which were thermally balanced to within 1·5 per cent, while the other example showed a heat balance deficiency exceeding 13 per cent.

The aim of the present work was twofold:—

(a) to show that results could be obtained accurately and reliably, by means of the Collins and Daws' method; and
(b) to extend the original analysis to include unequal heating and cooling periods.

The first aim has been achieved by selecting suitable finite difference forms for numerical integration, and by ensuring that infinite series arising in the computation are accurately summed. The second aim has been achieved by suitable substitution of variables, and this generalised form has been confirmed by a trial computation.

A few anomalous results obtained to date are indicated, and these are now the subject of further study.

For convenience, the regenerator structure is assumed to be reduced to an "equivalent single channel". This is merely a simpler hypothetical structure, whose mathematical definition is thermally identical with that of the real regenerator.

STRUCTURE OF THE EQUIVALENT SINGLE CHANNEL

The concept of the equivalent single channel is most conveniently explained in relation to a real regenerator of the form shown in Fig. 1(a). In this arrangement, a series of equally spaced, identical, solid walls forms a system of similar ducts. All wall surfaces are parallel and extend throughout the regenerator in the direction of fluid flow. The regenerator width, l, is large compared with the separation, a', of the walls. Consequently, no serious error is introduced by neglecting the areas of the other surfaces which complete the regenerator boundary. All physical quantities vary only in the flow direction, and are identical in each duct. A system of this type can be replaced mathematically by a single channel regenerator.

The equivalent regenerator is shown in Fig. 1(b). It is bounded on two sides by walls whose surfaces are all parallel. Additional surfaces, which complete the boundary, are of negligible area. In the direction of fluid flow, this regenerator has the same length, L, as the real regenerator. Also, in any plane perpendicular to the fluid flow, the real and equivalent systems have the same wetted perimeter, $2\bar{N}l$.

In the real regenerator, each wall is heated or cooled equally at both of its surfaces. Thus in any plane section perpendicular to the fluid flow, no heat flows across the mid-thickness of any wall. In the single channel equivalent, only one face of each wall is wetted, and so the wall surfaces remote from the duct are assumed to be insulated. In this case, the equivalent wall thickness is half the thickness of the real regenerator wall. The physical properties of equivalent walls are identical to those of the real walls.

Operating parameters (e.g. fluid flow rates, reversal times, and heat transfer coefficients) specified for the real regenerator, apply unchanged to the equivalent regenerator.

The concept of the equivalent single channel reduces the regenerator problem to one temporal and two spatial dimensions. Most calculation methods treat this simplified form of the problem. As seen above, for a simple filling of the type shown in Fig. 1(a), the equivalent single channel can be derived logically from physical considerations. For more complex geometries, some authors [3, 7] recommend that the equivalent single channel regenerator should have a wall thickness equal to the ratio-volume of solid filling to total wetted surface area. The real and equivalent regenerators should always have the same wetted surface area.

SIMPLIFYING ASSUMPTIONS

Several assumptions, common to most analyses of the regenerator problem, are listed below. Most of these are featured in the computed examples. However, the analysis will sometimes cope with the more general case, and when this is so will be clear for the subsequent text.

(a) The mass flow rate of the fluid is large enough to permit the use of the approximation

$$W_iS_i\frac{\partial t}{\partial y} + w_iS_i\frac{\partial t}{\partial \theta} \simeq W_iS_i\frac{\partial t}{\partial y} \tag{1}$$

(The left hand side is the expression for the rate at which the fluid absorbs heat from unit area of wall surface when gaseous conduction is assumed negligible.)

If this approximation is not sufficiently accurate, a suitable change of variable can achieve the form on the right hand side of the equation [12–14].

(b) Conduction through the solid walls is negligible in the direction of the gas flow. This assumption simplifies the conductivity equation to an unidirectional form. Practical considerations which justify this approximation are:

(i) Temperature gradients parallel to the fluid flow are usually much smaller than gradients normal to the wall surface; and

(ii) in many applications, the filling consists of unbonded courses of brick, and so there is a high thermal resistance between successive layers.

(c) The flow regime is sufficiently turbulent to achieve a uniform fluid temperature in any plane perpendicular to the direction of flow.

(d) Heating and cooling periods are of equal duration.

(e) Physical properties of the solid filling and of the hot and cold fluids are invariant both spatially and temporally.

(f) Wall dimensions are constant throughout the regenerator.

(g) Entry temperatures of the hot and cold fluids are constant in time.

(h) In both periods, the heat transfer rate is assumed to be proportional to the temperature difference between the solid surface and the adjacent fluid.

(i) Heat transfer coefficients are constant throughout each period, and do not vary along the regenerator.

MATHEMATICAL STATEMENT OF THE PROBLEM

Basic equations

Collins and Daws [10] stated the problem for equal heating and cooling periods. This particular form is stated in this section. The subsequent text as far as equation (36) follows that of Collins and Daws, except that some algebraic detail is omitted, and amendments have been introduced to achieve generality whenever possible.

The y-axis, parallel to the fluid flow, has its origin at the centre of the regenerator. Hot fluid enters the regenerator at $y = +(L/2)$, and cold fluid enters at $y = -(L/2)$.

The x-axis is perpendicular to the wall surfaces, and has its zero at one of those surfaces. Its positive direction is into the solid wall at the origin. Since the two walls bounding the duct are identical, and since the fluid temperature is independent of x, only positive x is relevant.

The commencement of an arbitarily selected heating period is taken as the origin of time.

The structure of the single channel regenerator and the assumption of zero conductivity parallel to the fluid flow reduce the conductivity equation to the unidirectional form

$$\alpha \frac{\partial^2 T(x, y, \theta)}{\partial x^2} = \frac{\partial T(x, y, \theta)}{\partial \theta}, \quad \text{for} \quad 0 \leqslant x \leqslant a, \quad -(L/2) \leqslant y \leqslant +(L/2), \quad \theta \geqslant 0. \tag{2}$$

Since the wall is impervious to heat at $x = a$,

$$\frac{\partial T(x, y, \theta)}{\partial x} = 0, \quad \text{for} \quad x = a, \quad \theta \geqslant 0. \tag{3}$$

At the wetted surface of the wall, for the heating period, the continuity and heat balance equations are respectively

$$\left.\begin{aligned} -k \frac{\partial T(x, y, \theta)}{\partial x} &= h_1(t - T) \\ W_1 S_1 \frac{\partial t_1}{\partial y} &= -kp \frac{\partial T}{\partial x} \end{aligned}\right\} \text{for} \quad 2\bar{n}P \leqslant \theta \leqslant (2\bar{n} + 1)P \quad \text{and} \quad x = 0. \tag{4}$$

The latter equation incorporates the approximation of equation (1).

Similarly, for the cooling period,

$$\left.\begin{aligned} -k \frac{\partial T}{\partial x} &= h_2(t - T) \\ W_2 S_2 \frac{\partial t_2}{\partial y} &= kp \frac{\partial T}{\partial x} \end{aligned}\right\} \text{for} \quad (2\bar{n} + 1)P \leqslant \theta \leqslant (2\bar{n} + 2)P \quad \text{and} \quad x = 0. \tag{5}$$

Taking the general case where the inlet temperatures of the hot and cold fluids are not constant in their respective periods, but nevertheless are known functions of time,

$$\left.\begin{aligned} &t(y, \theta) = t_1(y, \theta), \\ &\text{and in particular,} \\ &t[(L/2), \theta] = t_1[(L/2), \theta] \end{aligned}\right\} \text{for} \quad 2\bar{n}P \leqslant \theta \leqslant (2\bar{n} + 1)P. \tag{6}$$

Also,

$$\left.\begin{aligned} &t(y, \theta) = t_2(y, \theta), \\ &\text{and in particular,} \\ &t[-(L/2), \theta] = t_2[-(L/2), \theta] \end{aligned}\right\} \text{for} \quad (2\bar{n} + 1)P < \theta < (2\bar{n} + 2)P. \tag{7}$$

The problem is to determine $T(x, y, \theta)$, $t_1(y, \theta)$ and $t_2(y, \theta)$ for all y and all θ, when the regenerator operates in the condition of cyclic equilibrium.

Non-dimensional form of basic equations

When the mean fluid inlet temperatures are $\bar{t}_1(L/2)$ and $\bar{t}_2[-(L/2)]$, all temperatures are rendered dimensionless by the substitution:

$$U' = \frac{2U}{\bar{t}_1(L/2) - \bar{t}_2(-L/2)} - \frac{\bar{t}_1(L/2) + \bar{t}_2(-L/2)}{\bar{t}_1(L/2) - \bar{t}_2(-L/2)}. \tag{8}$$

This substitution, where U is t, t_1, t_2 or T, leaves equations (2)–(7) unchanged. The mean temperatures of the hot and cold fluids at the entry to the regenerator become

$$\bar{t}'_1(L/2) = +1 \quad \text{and} \quad \bar{t}'_2(-L/2) = -1 \text{ respectively.}$$

Now the following normalizing substitutions are made

$$\left.\begin{array}{ll} x = a\xi, & y = (L/2)\zeta, \quad \theta = (a^2 z/\alpha), \\ \dfrac{h_1 a}{k} = N_1, & \dfrac{h_2 a}{k} = N_2, \\ \dfrac{2W_1 S_1 a}{kpL} = M_1, & \dfrac{2W_2 S_2 a}{kpL} = M_2, \\ t'(y,\theta) = \chi(\zeta, z), & T'(x, y, \theta) = \Xi(\xi, \zeta, z), \\ t'_1[(L/2), \theta] = \chi_1(1, z), \quad \text{and} & t'_2[-(L/2), \theta] = \chi'_2(-1, z). \end{array}\right\} \tag{9}$$

At entry to the regenerator, the mean gas temperatures during the heating and cooling periods now become respectively

$$\left.\begin{array}{l} \bar{\chi}_1(1, z) = 1, \quad \text{and} \\ \bar{\chi}_2(-1, z) = -1 \end{array}\right\} \tag{10}$$

As a result of the substitutions (9), equations (2)–(7) become

$$\frac{\partial^2 \Xi}{\partial \xi^2} = \frac{\partial \Xi}{\partial z}, \quad \text{for} \quad 0 \leqslant \xi \leqslant 1, -1 \leqslant \zeta \leqslant +1 \quad \text{and} \quad z \geqslant 0 \tag{11}$$

$$\frac{\partial \Xi}{\partial \xi} = 0, \quad \text{for} \quad \xi = 1 \quad \text{and} \quad z \geqslant 0 \tag{12}$$

$$\frac{\partial \Xi}{\partial \xi} = N_1(\Xi - \chi) = -M_1 \frac{\partial \chi}{\partial \zeta}, \quad \text{for} \quad 2\bar{n}\Omega \leqslant z \leqslant (2\bar{n} + 1)\Omega, \quad \text{and} \quad \xi = 0 \tag{13}$$

$$\frac{\partial \Xi}{\partial \xi} = N_2(\Xi - \chi) = M_2 \frac{\partial \chi}{\partial \zeta}, \quad \text{for} \quad (2\bar{n} + 1)\Omega \leqslant z \leqslant (2\bar{n} + 2)\Omega, \quad \text{and} \quad \xi = 0 \tag{14}$$

$$\left.\begin{array}{l} \chi(\zeta, z) = \chi_1(\zeta, z), \\ \text{and in particular} \\ \chi(1, z) = \chi_1(1, z) \end{array}\right\} \quad \text{for} \quad 2\bar{n}\Omega \leqslant z \leqslant (2\bar{n} + 1)\Omega \tag{15}$$

$$\left.\begin{aligned}&\chi(\zeta, z) = \chi_2(\zeta, z),\\ &\text{and in particular}\\ &\chi(-1, z) = \chi_2(-1, z)\end{aligned}\right\} \text{for} \quad (2\bar{n}+1)\Omega \leqslant z \leqslant (2\bar{n}+2)\Omega \tag{16}$$

In the above,

$$\Omega = \frac{\alpha P}{a^2}. \tag{17}$$

In this form, with $\chi_1(1, z)$ given for $0 \leqslant z \leqslant \Omega$, and $\chi_2(-1, z)$ given for $\Omega \leqslant z \leqslant 2\Omega$, the problem is to determine all Ξ, $\chi_1(\zeta, z)$ and $\chi_2(\zeta, z)$.

COLLINS AND DAWS' SOLUTION

Collins and Daws derived a solution for equation (11) and then imposed the boundary condition of equation (12), together with the reversal condition [10]. The following result was obtained

$$\Xi(0, \zeta, \tau) + \int_0^\tau \left[\frac{\partial \Xi(0, \zeta, z)}{\partial \xi} - \Xi(0, \zeta, z)\right] \sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(\tau - z)\right] \mathrm{d}z$$
$$+ \int_0^{2\Omega} \left[\frac{\partial \Xi(0, \zeta, z)}{\partial \xi} - \Xi(0, \zeta, z)\right] \sum_{s=1}^{\infty} \frac{b_s \exp\left[-\beta_s(\tau + 2\Omega - z)\right]}{1 - \exp(-2\Omega\beta_s)} \mathrm{d}z = 0. \tag{18*}$$

In this equation,

$$b_s = \frac{2\beta_s}{2 + \beta_s} \tag{19}$$

where β_s is the sth root of the equation

$$\sqrt{\beta} \tan \sqrt{\beta} = 1. \tag{20}$$

$\Xi(0, \zeta, z)$ and $[\partial \Xi(0, \xi, z)/\partial \xi]$ are related to the fluid temperature and its gradient in the direction of fluid flow by equations (13) and (14). Substituting these into equation (18) and then applying the result at times τ and $\tau + \Omega$, where the origin of time is taken at the beginning of a heating period, equations (21) and (22) are obtained as follows

$$a_1 f_1(\zeta, \tau) + a_2 f_2(\zeta, \tau) + a_4 X_1(\zeta, \tau) + a_3 X_2(\zeta, \tau)$$
$$+ 2 \int_{z'=0}^{\tau} X_1(\zeta, z')\eta(\tau - z')\,\mathrm{d}z' + 2 \int_{z'=0}^{\Omega} X_1(\zeta, z')\,\eta_1(\tau - z')\,\mathrm{d}z' = 0 \tag{21}$$

and,

$$-a_1 f_1(\zeta, \tau) + a_2 f_2(\zeta, \tau) + a_3 X_1(\zeta, \tau) + a_4 X_2(\zeta, \tau) + 2 \int_{z'=0}^{\tau} X_2(\zeta, z')\eta(\tau - z')\,\mathrm{d}z'$$
$$- 2 \int_{z'=0}^{\Omega} X_2(\zeta, z')\,\eta_2(\tau - z')\,\mathrm{d}z' = 0 \tag{22}$$

* The derivation of this equation is given in the Appendix.

where

$$-\left[M_1\left(1-\frac{1}{N_1}\right)\frac{\partial f_1}{\partial\zeta}+f_1\right]+\left[M_2\left(1-\frac{1}{N_2}\right)\frac{\partial f_2}{\partial\zeta}-f_2\right]=2X_1 \tag{23}$$

$$\left[M_1\left(1-\frac{1}{N_1}\right)\frac{\partial f_1}{\partial\zeta}+f_1\right]+\left[M_2\left(1-\frac{1}{N_2}\right)\frac{\partial f_2}{\partial\zeta}-f_2\right]=2X_2 \tag{24}$$

$$\eta(\tau)=\sum_{s=1}^{\infty} b_s\exp(-\beta_s\tau) \tag{25}$$

$$\eta_1(\tau)=\sum_{s=1}^{\infty}\frac{b_s\exp[-\beta_s(\tau+\Omega)]}{[1-\exp(-\beta_s\Omega)]} \tag{26}$$

$$\eta_2(\tau)=\sum_{s=1}^{\infty}\frac{b_s\exp[-\beta_s(\tau+\Omega)]}{[1+\exp(-\beta\Omega)]} \tag{27}$$

$$\left.\begin{array}{l} z'=0 \text{ at the commencement of any heating period}\\ z'=z, \quad \text{for} \quad 0<z<\Omega\\ z'=z-\Omega, \quad \text{for} \quad \Omega<z<2\Omega\\ \text{and } \chi_i(\zeta,z) \text{ transforms to } f_i(\zeta,z') \end{array}\right\} \tag{28}$$

Equations (21) and (22) have to be solved knowing

$$f_1(1,\tau) \quad \text{and} \quad f_2(-1,\tau).$$

Collins and Daws then considered $(n+1)$ points dividing the range of z' into n equal intervals, so that at any particular value of ζ, (21) and (22) could be written approximately

$$a_1F_1+a_2F_2+a_3X_2'+A_1X_1'=0 \tag{29}$$

$$-a_1F_1+a_2F_2+a_3X_1'+A_2X_2'=0 \tag{30}$$

where F_1, F_2, X_1' and X_2', replacing f_1, f_2, X_1 and X_2 of (21) and (22), denote column matrices covering all values of τ, and A_1 and A_2 are square $(n+1)\times(n+1)$ matrices. The elements of A_1 and A_2 depend on the particular finite difference forms selected to approximate the definite integrals occurring in (21) and (22). In the general case A_1 and A_2 would also depend on ζ.

The equations (29) and (30) can ultimately be arranged as

$$\frac{\partial F_2}{\partial\zeta}+\gamma_1F_2+\gamma_2F_1=0 \tag{31}$$

$$\frac{\partial F_1}{\partial\zeta}+T_1F_1+T_2F_2=0 \tag{32}$$

where

$$\left.\begin{aligned}\gamma_1 &= \frac{[(A_2A_1 - a_3^2I)^{-1}\,a_2(A_2 - a_3I) + (A_1A_2 - a_3^2I)^{-1}\,a_2(A_1 - a_3I) - I]}{M_2[1 - (1/N_2)]}\\ \gamma_2 &= \frac{[(A_2A_1 - a_3^2I)^{-1}\,a_1(a_3I + A_2) - (A_1A_2 - a_3^2I)^{-1}\,a_1(A_1 + a_3I)]}{M_2[1 - (1/N_2)]}\end{aligned}\right\} \quad (33)$$

$$\left.\begin{aligned}T_1 &= -\frac{[(A_2A_1 - a_3^2I)^{-1}\,a_1(a_3I + A_2) + (A_1A_2 - a_3^2I)^{-1}\,a_1(A_1 + a_3I) - I]}{M_1[1 - (1/N_1)]}\\ \text{and}\quad T_2 &= -\frac{[(A_2A_1 - a_3^2I)^{-1}\,a_2(A_2 - a_3I) - (A_1A_2 - a_3^2I)^{-1}\,a_2(A_1 - a_3I)]}{M_1[1 - (1/N_1)]}\end{aligned}\right\} \quad (34)$$

Equations (31) and (32) apply at all points along the regenerator, and at each value of ζ defining the boundary of an element for numerical integration. If the physical properties are assumed to be invariant then equations (31) and (32) remain numerically identical along the regenerator.

COMPUTATION PROCESS

General considerations

Computation, using the Collins and Daws' technique, is conveniently considered in three stages:

(i) application of numerical approximations to convert equations (21) and (22) to the form of (29) and (30);
(ii) matrix arithmetic [see equations (33) and (34)] to obtain the matrices γ_1, γ_2, T_1 and T_2 of equations (31) and (32); and
(iii) integration of (31) and (32) along the regenerator, with fluid entry temperatures known as boundary conditions.

Considering stage (i), we note that the matrices:

$$A_1' = A_1 - a_4I \quad (35)$$

and

$$A_2' = A_2 - a_4I \quad (36)$$

are functions only of Ω and of the approximate forms used for integrating with respect to time.

The computation was developed using a Ferranti Pegasus digital computer, and programming was divided into two separate parts. The first was in Autocode, its function being that of (i) above; i.e. determination of the matrices A_1' and A_2' of equations (35) and (36) by some process for numerical integration with respect to time. The second part of the computation was programmed in machine code—its purpose being to compute stages (ii) and (iii). This programme required M_1, M_2, N_1, N_2, A_1' and A_2' as input data, and completed the computation by integrating numerically along the regenerator.

Dividing the computation in this way offered the advantage that any number of examples could be computed for a single value of Ω, without repeating the computation of A_1' and A_2'. A further advantage was that the programming effort required for stage (i) was minimised by using the comparatively simple Autocode. This is particularly advantageous for exploring possible computation techniques, as was being done in this case.

Considering equations (21), (29) and (35) it will be seen that in order to establish the matrix A'_1, we have to represent

$$I_1 = 2\left[\int_0^{(\Omega r/n)} X_1(\zeta, z').\,\eta\left(\frac{\Omega r}{n} - z'\right) dz' + \int_0^{\Omega} X_1(\zeta, z').\,\eta_1\left(\frac{\Omega r}{n} - z'\right) dz'\right] \tag{37}$$

in the form

$$I_1 \simeq \sum_{s=0}^{n} X_{1s}\rho_{1rs} \tag{38}$$

where

$$X_{1s} \text{ is } X_1\left(\zeta, \frac{\Omega s}{n}\right).$$

This must be done for all integral values of r in the range $0 \leqslant r \leqslant n$, where equations (21) and (22) are applied at discrete values of τ given by $\tau = (\Omega r/n)$.

Since

$$\eta\left(\frac{\Omega r}{n} - z'\right) + \eta_1\left(\frac{\Omega r}{n} - z'\right) = \eta_1\left(\frac{\Omega r}{n} - \Omega - z'\right) \tag{39}$$

equation (37) becomes

$$I_1 = 2\left[\int_0^{(\Omega r/n)} X_1(\zeta, z').\,\eta_1\left(\frac{\Omega r}{n} - \Omega - z'\right) dz' + \int_{(\Omega r/n)}^{\Omega} X_1(\zeta, z').\,\eta_1\left(\frac{\Omega r}{n} - z'\right) dz'\right]. \tag{40}$$

Similarly, to establish the matrix A'_2 we have to represent

$$I_2 = 2\left[\int_0^{(\Omega r/n)} X_2(\zeta, z').\,\eta\left(\frac{\Omega r}{n} - z'\right) dz' - \int_0^{\Omega} X_2(\zeta, z').\,\eta_2\left(\frac{\Omega r}{n} - z'\right) dz'\right] \tag{41}$$

in the form

$$I_2 \simeq \sum_{s=0}^{n} X_{2s}\rho_{2rs} \tag{42}$$

where X_{2s} is $X_2[\zeta, (\Omega_s/n)]$. Again, this must be done for all integral r in the range $0 \leqslant r \leqslant n$. Also

$$\eta\left(\frac{\Omega r}{n} - z'\right) - \eta_2\left(\frac{\Omega r}{n} - z'\right) = \eta_2\left(\frac{\Omega r}{n} - \Omega - z'\right) \tag{43}$$

so that equation (41) becomes

$$I_2 = 2\left[\int_0^{(\Omega r/n)} X_2(\zeta, z').\,\eta_2\left(\frac{\Omega r}{n} - \Omega - z'\right) dz' - \int_{(\Omega r/n)}^{\Omega} X_2(\zeta, z').\,\eta_2\left(\frac{\Omega r}{n} - z'\right) dz'\right]. \tag{44}$$

It is assumed that n is large enough so that $X_i(\zeta, z')$ (where $i = 1$ or 2) is approximately linear throughout the range $(\Omega r/n) - h \leqslant z' \leqslant (\Omega r'/n)$, where r is integral, and in the range $1 \leqslant r' \leqslant n$.

For any such interval we may then write

$$X_i(\zeta, z') = Az' + B \tag{45}$$

where A and B will depend on ζ and r'.

For any such interval in the numerical computation of equations (40) or (44), we put

$$U_i(C, z') = \int \eta_i(C - z')\,dz' \tag{46}$$

where $C = \tau - \Omega$ for $z' \leqslant \tau$ and $C = \tau$ for $z' \geqslant \tau$.

We obtain on integrating by parts,

$$\tfrac{1}{2}\delta I_i = \int_{(\Omega r'/n)-h}^{(\Omega r'/n)} X_i \,.\, \eta_i(C - z')\,.\,dz' = \Big[X_i(\zeta, z')\,.\,U_i(C, z')\Big]_{(\Omega r'/n)-h}^{(\Omega r'/n)} - \int_{(\Omega r'/n)-h}^{(\Omega r'/n)} \frac{dX_i(\zeta, z')}{dz'}\,.\,U_i(C, z')\,.\,dz'.$$

Applying equation (45) to the second term, we obtain

$$\tfrac{1}{2}\delta I_i = \Big[X_i(\zeta, z')\,.\,U_i(C, z')\Big]_{(\Omega r'/n)-h}^{(\Omega r'/n)} - A\int_{(\Omega r'/n)-h}^{(\Omega r'/n)} U_i(C, z')\,dz'. \tag{47}$$

From equation (45),

$$A = \frac{1}{h}\left[X_i\left(\zeta, \frac{\Omega r'}{n}\right) - X_i\left(\zeta, \frac{\Omega r'}{n} - h\right)\right],$$

and so equation (47) becomes

$$\tfrac{1}{2}\delta I_i = \Big[X_i(\zeta, z')\,.\,U_i(C, z')\Big]_{(\Omega r'/n)-h}^{(\Omega r'/n)} - \left[X_i\left(\zeta, \frac{\Omega r'}{n}\right) - \left[X_i\left(\zeta, \frac{\Omega r'}{n} - h\right)\right]\frac{1}{h}\int_{(\Omega r'/n)-h}^{(\Omega r'/n)} U_i(C, z')\,dz'.\right] \tag{48}$$

Writing

$$\overline{U}_i\left(\frac{\Omega r'}{n}, C\right) = \frac{1}{h}\int_{(\Omega r'/n)-h}^{(\Omega r'/n)} U_i(C, z')\,dz' \tag{49}$$

we can rearrange equation (48) to become

$$\tfrac{1}{2}\delta I_i = X_i\left(\zeta, \frac{\Omega r'}{n} - h\right)\left[\overline{U}_i\left(\frac{\Omega r'}{n}, C\right) - U_i\left(C, \frac{\Omega r'}{n} - h\right)\right] - X_i\left(\zeta, \frac{\Omega r'}{n}\right)\left[\overline{U}_i\left(\frac{\Omega r'}{n}, C\right) - U_i\left(C, \frac{\Omega r'}{n}\right)\right]. \tag{50}$$

For $i = 1$ or 2, we have from equations (26), (27) and (46)

$$U_i(C, z') = \int \sum_{s=1}^{\infty} \frac{b_s \exp\left[-\beta_s(C + \Omega - z')\right]}{1 + (-1)^i \exp(-\beta_s\Omega)}\,dz' = \sum_{s=1}^{\infty} \frac{b_s}{\beta_s} \frac{\exp\left[-\beta_s(C + \Omega - z')\right]}{1 + (-1)^i \exp(-\beta_s\Omega)}. \tag{51}$$

Substituting this in equation (49), we have

$$\overline{U}_i\left(\frac{\Omega r'}{n}, c\right) = \sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp\{-\beta_s[C + \Omega - (\Omega r'/n)]\}}{1 + (-1)^i \exp(-\beta_s \Omega)} [1 - \exp(-h\beta_s)]. \tag{52}$$

With $i = 1$, using equations (51) and (52) in equation (50) and then applying $r = 0, 1, 2 \ldots n$, successively, in equation (40), we obtain

$$A'_1 = \begin{bmatrix} H_1(\Omega), & H_2[(n-1)h], & H_2[(n-2)h], & H_2[(n-3)h], & H_2(h), & H_3(h) \\ H_1(h), & H_3(h)+H_1(\Omega), & H_2[(n-1)h], & H_2[(n-2)h], & H_2(2h), & H_3(2h) \\ H_1(2h), & H_2(h), & H_3(h)+H_1(\Omega), & H_2[(n-1)h], & & \\ H_1(3h), & H_2(2h), & H_2(h), & H_3(h)+H_1(\Omega), & & \\ & & & & & \\ H_1[(n-1)h], & H_2[(n-2)h], & & & H_3(h) + H_1(\Omega), & H_3(\Omega) \\ H_1(\Omega), & H_2[(n-1)h], & H_2[(n-2)h], & & H_2(h), & H_3(h) \end{bmatrix} \tag{53}$$

where

$$H_1(Nh) = 2\sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp(-Nh\beta_s)}{[1 - \exp(-\Omega\beta_s)]} [\exp(h\beta_s) - 1 - h\beta_s], \text{ for } 1 \leqslant N \leqslant n \tag{54}$$

$$H_2(Nh) = 2\sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp[-(N+1)h\beta_s]}{[1 - \exp(-\Omega\beta_s)]} [\exp(h\beta_s) - 1]^2 \text{ for } 1 \leqslant N \leqslant n - 1 \tag{55}$$

and

$$H_3(Nh) = 2\sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp(-Nh\beta_s)}{[1 - \exp(-\Omega\beta_s)]} [h\beta_s \exp(h\beta_s) - \exp(h\beta_s) + 1] \text{ for } 1 \leqslant N \leqslant n. \tag{56}$$

Again, with $i = 2$, using equations (51) and (52) in (50) and then applying to equation (44) for $r = 0, 1, 2, \ldots n$, successively, we obtain

$$A'_2 = \begin{bmatrix} -H_4(\Omega), & -H_5[(n-1)h], & -H_5[(n-2)h], & -H_5(2h), & -H_6(2h) \\ H_4(h), & H_6(h) - H_4(\Omega), & -H_5[(n-1)h], & -H_5(h), & -H_6(h) \\ H_4(2h), & H_5(h), & H_6(h) - H_4(\Omega), & & \\ & H_5(2h), & H_5(h), & & \\ & & & & \\ & & & -H_5[(n-1)h], & -H_6[(n-1)h] \\ H_4[(n-1)h], & H_5[(n-2)h], & H_5[(n-3)h], & H_6(h) - H_4(\Omega), & -H_6(\Omega) \\ H_4(\Omega), & H_5[(n-1)h], & H_5[(n-2)h], & H_5(h), & H_6(h) \end{bmatrix} \tag{57}$$

where

$$H_4(Nh) = 2\sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp(-Nh\beta_s)}{[1+\exp(-\Omega\beta_s)]} [\exp(h\beta_s) - 1 - h\beta_s], \text{ for } 1 \leqslant N \leqslant n \tag{58}$$

$$H_5(Nh) = 2\sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp[-(N+1)h\beta_s]}{[1+\exp(-\Omega\beta_s)]} [\exp(h\beta_s) - 1]^2, \text{ for } 1 \leqslant N \leqslant n-1 \tag{59}$$

and

$$H_6(Nh) = 2\sum_{s=1}^{\infty} \frac{b_s}{h\beta_s^2} \frac{\exp(-Nh\beta_s)}{[1+\exp(-\Omega\beta_s)]} \cdot [h\beta_s \exp(h\beta_s) - \exp(h\beta_s) + 1], \text{ for } 1 \leqslant N \leqslant n. \tag{60}$$

A number of relationships were derived to enable all the matrix elements of (53) and (57) to be computed to any preselected accuracy [8]. In the worked examples these were always computed to within $\pm 0{\cdot}1$ per cent.

Integration along the regenerator

After the matrices A_1' and A_2' have been calculated, it becomes possible to calculate the matrices γ_1, γ_2, T_1 and T_2 of equations (31) and (32). The latter can be combined into a single differential equation

$$\frac{\partial F'}{\partial \zeta} + P'F' = 0 \tag{61}$$

where F' is the column matrix $\{F_2, F_1\}$ and P' is a square matrix defined by

$$P' = \left[\begin{array}{c:c} \gamma_1 & \gamma_2 \\ \hdashline T_2 & T_1 \end{array}\right]. \tag{62}$$

An equation of the type (61) can be applied at all positions along the regenerator, and in particular at each of the $(2m+1)$ points which divide the regenerator into $2m$ equal parts. If we now number these points from zero at midway along the regenerator to $-m$ at the cold end and $+m$ at the hot end, then $\zeta = (r/m)$ at the point numbered r. At this point, suppose that F' and P' of equation (61) have values F_r' and P_r'. Then, for each interval along the regenerator, we replace equation (61) by the approximation

$$\frac{F'_{r+1} - F'_r}{1/m} + \tfrac{1}{2}(P'_{r+1}F'_{r+1} + P'_rF'_r) = 0,$$

or

$$F'_{r+1} = (2mI + P'_{r+1})^{-1}(2mI - P'_r)F'_r. \tag{63}$$

When physical properties are invariant along the regenerator this becomes

$$F'_{r+1} = (2mI + P')^{-1}(2mI - P')F'_r. \tag{64}$$

Knowing any F_r', the regenerator problem can be solved completely by repeated application of this recurrence relation (64). However, the known boundary conditions are the entry temperatures of the two fluids and so for a counter-flow heat exchanger no F'_r is initially known completely. However, $F_{2,-m}$ and $F_{1,+m}$ are specified completely, and in particular are known in each period at the $(n+1)$ points which divide the range of Ω into n equal parts. Illustrating this, for $n = 5$ and constant fluid entry temperatures, we thus have

$$F_{2,-m} = \{-1, -1, -1, -1, -1, -1] \tag{65}$$

and

$$F_{1,+m} = \{+1, +1, +1, +1, +1, +1\}. \tag{66}$$

We must now solve relation (64) [or (63) in the more general case] along the regenerator for the boundary conditions (65) and (66).

A total of $(n+2)$ solutions are first obtained, each for a particular hypothetical boundary condition. Illustrating this for $n = 5$, seven hypothetical boundary states are defined for F'_{-m} as follows:

$$F'^{(1)}_{-m} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad F'^{(2)}_{-m} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \ldots F'^{(6)}_{-m} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{bmatrix}, \quad F'^{(7)}_{-m} = \begin{bmatrix} -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \\ -1 \end{bmatrix} \tag{67}$$

At a position $\zeta = (r/m)$ along the regenerator, the true solution F'_r is given by

$$F'_r = K_1 F'^{(1)}_r + K_2 F'^{(2)}_r + \ldots + K_{n+2} F'^{(n+2)}_r \tag{68}$$

where $F'^{(s)}_{(r)}$ is the solution for F'_r which satisfies the hypothetical boundary state $F'^{(s)}_{-m}$, defined in (67) for the special case where $n = 5$.

In equation (68), K_1 to K_{n+2} are coefficients which do not vary the regenerator, i.e. they are independent of r. These constants can be determined by applying equation (68) at $r = -m$ and $r = +m$, so as to satisfy equations (65) and (66).

Equation (65) gives only one relationship, namely

$$\sum_{s=1}^{n+2} K_s = +1. \tag{69}$$

Equation (66) gives the $(n+1)$ equations

$$\sum_{s=1}^{n+2} F'^{(s)}_{m,j} K_s = +1, \text{ where } n+2 \leqslant j \leqslant 2n+2. \tag{70}$$

Here $F'^{(s)}_{m,j}$ is the jth element of the column matrix $F'^{(s)}_m$. Thus (69) and (70) together give $(n+2)$

equations, which can be solved for the $(n + 2)$ unknown constants K. This is most conveniently done using matrix algebra.

The problem is now solved completely, since the calculated constants K can be applied to equation (68) for $r = -m$ after which equation (63) can be applied successively along the regenerator.

DEVELOPMENT FOR UNEQUAL HEATING AND COOLING PERIODS

If Ω has values Ω_1 and Ω_2 for the heating and cooling periods respectively, then the following, more general form, must be used in place of equation (18)

$$\Xi(0,\zeta,\tau) + \int_0^{\tau}\left[\frac{\partial\Xi(0,\zeta,z)}{\partial\xi} - \Xi(0,\zeta,z)\right]\sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(\tau - z)\right] dz$$
$$+ \int_0^{\Omega_1+\Omega_2}\left[\frac{\partial\Xi(0,\zeta,z)}{\partial\xi} - \Xi(0,\zeta,z)\right]\sum_{s=1}^{\infty}\frac{b_s \exp\left[-\beta_s(\tau + \Omega_1 + \Omega_2 - z)\right]}{1 - \exp\left[-\beta_s(\Omega_1 + \Omega_2)\right]} dz = 0. \qquad (71)^*$$

We need only consider one cycle, and so choose for convenience the range $0 \leqslant z \leqslant \Omega_1 + \Omega_2$ where $z = 0$ commences a heating period.

We now introduce a time v defined by

$$\left.\begin{aligned} v &= \frac{z}{\Omega_1}, \quad \text{for} \quad 0 < z < \Omega_1 \\ v &= \frac{z - \Omega_1}{\Omega_2}, \quad \text{for} \quad \Omega_1 < z \leqslant \Omega_1 + \Omega_2. \end{aligned}\right\} \qquad (72)$$

In this case, $\chi(\zeta, z)$ transforms as follows

$$\left.\begin{aligned} &\chi(\zeta, z) = f_1(\zeta, v), \quad \text{for} \quad 0 \leqslant z \leqslant \Omega_1. \\ \text{and} \quad &\chi(\zeta, z) = f_2(\zeta, v), \quad \text{for} \quad \Omega_1 \leqslant z \leqslant \Omega_1 + \Omega_2. \end{aligned}\right\} \qquad (73)$$

Equations (13) and (14) apply, but with the revised time scale we now have

$$\frac{\partial\Xi}{\partial\xi} = N_1(\Xi - \chi) = -M_1\frac{\partial\chi}{\partial\zeta}, \quad \text{for} \quad \xi = 0, \quad 0 \leqslant z \leqslant \Omega_1 \qquad (74)$$

and

$$\frac{\partial\Xi}{\partial\xi} = N_2(\Xi - \chi) = M_2\frac{\partial\chi}{\partial\zeta}, \quad \text{for} \quad \xi = 0, \quad \Omega_1 \leqslant z \leqslant \Omega_1 + \Omega_2. \qquad (75)$$

We now use equations (74) and (75) to substitute for $\Xi(0, \zeta, z)$ and $[\partial\Xi(0, \zeta, z)/\partial\xi]$ in terms of χ and $(\partial\chi/\partial\zeta)$ in equation (71), and apply the result first at $z = \tau$ and then at $z = [(\Omega_2/\Omega_1)\tau + \Omega_1]$ (where $\tau \leqslant \Omega_1$). This process, together with the relations (73), give the two equations

$$\frac{N_1}{N_1 - 1} f_1(\zeta, \bar{\tau}) + \frac{1}{\Omega_1(N_1 - 1)}\phi_1(\zeta, \bar{\tau}) + \int_{v=0}^{\bar{\tau}} \phi_1(\zeta, v) \, . \, g[\Omega_1(\bar{\tau} - v)] \, dv$$

* The derivation of this equation is given in the Appendix.

$$+ \int_{v=\bar{\tau}}^{1} \phi_1(\zeta, v) \, . \, g\left[\Omega_1\left(\bar{\tau} + 1 + \frac{\Omega_2}{\Omega_1} - v\right)\right] \mathrm{d}v + \int_{v=0}^{1} \phi_2(\zeta, v) \, . \, g\left[\Omega_2\left(\frac{\Omega_1}{\Omega_2}\bar{\tau} + 1 - v\right)\right] \mathrm{d}v = 0 \quad (76)$$

and

$$\frac{N_2}{N_2 - 1} f_2(\zeta, \bar{\tau}) + \frac{1}{\Omega_2(N_2 - 1)} \phi_2(\zeta, \tau) + \int_{v=0}^{\bar{\tau}} \phi_2(\zeta, v) \, . \, g[\Omega_2(\bar{\tau} - v)] \, \mathrm{d}v$$

$$+ \int_{v=\bar{\tau}}^{1} \phi_2(\zeta, v) \, . \, g\left[\Omega_2\left(\bar{\tau} + 1 + \frac{\Omega_1}{\Omega_2} - v\right)\right] \mathrm{d}v + \int_{v=0}^{1} \phi_1(\zeta, v) \, . \, g\left[\Omega_1\left(\frac{\Omega_1}{\Omega_2}\bar{\tau} + 1 - v\right)\right] \mathrm{d}v = 0 \quad (77)$$

where

$$\bar{\tau} = \frac{\tau}{\Omega_1} \quad (78)$$

$$\phi_1 = \Omega_1\left[-M_1\left(1 - \frac{1}{N_1}\right)\frac{\partial f_1}{\partial \zeta} - f_1\right] \quad (79)$$

$$\phi_2 = \Omega_2\left[M_2\left(1 - \frac{1}{N_2}\right)\frac{\partial f_2}{\partial \zeta} - f_2\right] \quad (80)$$

and

$$g(v) = \sum_{s=1}^{\infty} \frac{b_s \exp(-\beta_s v)}{1 - \exp[-\beta_s(\Omega_1 + \Omega_2)]}. \quad (81)$$

In these equations, v has its zero at the commencement of the heating and cooling periods. Also, in each period, v increases linearly with time to unity at the end of a heating or cooling period. For any $v = \bar{\tau}$, where $0 \leqslant \bar{\tau} \leqslant 1$, therefore, there are corresponding points in the heating and cooling periods, occurring at equal proportions of the total relevant duration.

As well as allowing Ω_1 and Ω_2 to be unequal, these more general equations also permit variation of brick properties along the regenerator. This advantage is obtained because at all points of the regenerator the parameter, v, varies linearly with time between the values of zero and unity.

We now have to solve equations (76) and (77), knowing $f_1(1, \bar{\tau})$ and $f_2(-1, \bar{\tau})$.

Divide the range of $\bar{\tau}$ into n equal intervals by $(n + 1)$ points in each period. Using approximate forms for the definite integrals, apply equations (76) and (77), at each of these points. We then have, at each value of ζ,

$$a_1 F_1 + (B_1 + d_1 I)\phi_1' + D_1 \phi_2' = 0 \quad (82)$$

and

$$a_2 F_2 + (B_2 + d_2 I)\phi_2' + D_2 \phi_1' = 0 \quad (83)$$

where F_1, F_2, ϕ_1' and ϕ_2', replacing f_1, f_2, ϕ_1 and ϕ_2 of equations (76) and (77), denote column matrices covering values at each discrete level of $\bar{\tau}$, and B_1, B_2, D_1 and D_2 are square $(n + 1) \times (n + 1)$ matrices whose elements will depend on the approximations used for the definite integrals occurring in (76) and (77).

Also,

$$\left.\begin{aligned} a_1 &= \frac{N_1}{N_1 - 1}, \qquad d_1 = \frac{1}{\Omega_1(N_1 - 1)} \\ a_2 &= \frac{N_2}{N_2 - 1}, \qquad d_2 = \frac{1}{\Omega_2(N_2 - 1)} \end{aligned}\right\}. \tag{84}$$

In the completely general case, B_1, B_2, D_1 and D_2 will depend on ζ, while a_1, a_2, d_1 and d_2 will depend on both ζ and $\bar{\tau}$. Eliminating first ϕ_2' and then ϕ_1' between equations (82) and (83) we obtain

$$a_1F_1 - a_2D_1(B_2 + d_2I)^{-1}F_2 + [(B_1 + d_1I) - D_1(B_2 + d_2I)^{-1}D_2]\phi_1' = 0 \tag{85}$$

and

$$a_2F_2 - a_1D_2(B_1 + d_1I)^{-1}F_1 + [(B_2 + d_2I) - D_2(B_1 + d_1I)^{-1}D_1]\phi_2' = 0. \tag{86}$$

Inverting the matrices $[(B_1 + d_1I) - D_1(B_2 + d_2I)^{-1}D_2]$ and $[(B_2 + d_2I) - D_2(B_1 + d_1I)^{-1}D_1]$ from equations (85) and (86) respectively, and then substituting for ϕ_1' and ϕ_2' from equations (79) and (80) we obtain

$$\frac{\partial F_1}{\partial \zeta} + V_1F_1 + V_2F_2 = 0 \tag{87}$$

and

$$\frac{\partial F_2}{\partial \zeta} + Z_1F_2 + Z_2F_1 = 0 \tag{88}$$

where

$$V_1 = \frac{-\{a_1^2[(B_1 + d_1I) - D_1(B_2 + d_2I)^{-1}D_2]^{-1} - \Omega_1a_1I\}}{\Omega_1M_1} \tag{89}$$

$$V_2 = \frac{\{a_1a_2[(B_2 + d_2I)D_1^{-1}(B_1 + d_1I) - D_2]^{-1}\}}{\Omega_1M_1} \tag{90}$$

$$Z_1 = \frac{\{a_2^2[(B_2 + d_2I) - D_2(B_1 + d_1I)^{-1}D_1]^{-1} - a_2\Omega_2I\}}{\Omega_2M_2} \tag{91}$$

and

$$Z_2 = \frac{-a_1a_2\{(B_1 + d_1I)D_2^{-1}(B_2 + d_2I) - D_1^{-1}\}}{\Omega_2M_2}. \tag{92}$$

Equations (87) and (88) now have the same form as equations (31) and (32) and so may be solved by any method suitable for the solution of (31) and (32).

Integration of the general equations in time

In equations (76) and (77), if $(n + 1)$ points divide the range of v into n equal parts and if n is large enough to allow the approximation that ϕ_1 and ϕ_2 are linear in the range

$$\frac{r}{n} \leqslant v \leqslant \frac{r+1}{n}$$

for all integral r, $0 \leqslant r \leqslant n - 1$, then by a method similar to that described earlier for numerical integration of equations (40) and (44) with respect to time, we obtain for the matrices B_1, B_2, D_1 and D_2.

$B_i =$

$$\begin{bmatrix} Q_i\left(1+\frac{\Omega_j}{\Omega_i}\right), & Q_i\left(\frac{n-1}{n}+\frac{\Omega_j}{\Omega_i}\right)+R_i\left(\frac{n-1}{n}+\frac{\Omega_j}{\Omega_i}\right), & & \cdot\ , & Q_i\left(\frac{1}{n}+\frac{\Omega_j}{\Omega_i}\right)+R_i\left(\frac{1}{n}+\frac{\Omega_j}{\Omega_i}\right), & R_i\left(\frac{\Omega_j}{\Omega_i}\right) \\ Q_i\left(\frac{1}{n}\right), & R_i(0)+Q_i\left(1+\frac{\Omega_j}{\Omega_i}\right), & Q_i\left(\frac{n-1}{n}+\frac{\Omega_j}{\Omega_i}\right)+R_i\left(\frac{n-1}{n}+\frac{\Omega_j}{\Omega_i}\right), & , & \cdot\ , & R_i\left(\frac{1}{n}+\frac{\Omega_j}{\Omega_i}\right) \\ Q_i\left(\frac{2}{n}\right), & Q_i\left(\frac{1}{n}\right)+R_i\left(\frac{1}{n}\right), & R_i(0)+Q_i\left(1+\frac{\Omega_j}{\Omega_i}\right), & \cdot\ , & & R_i\left(\frac{2}{n}+\frac{\Omega_j}{\Omega_i}\right) \\ Q_i\left(\frac{n-1}{n}\right), & Q_i\left(\frac{n-2}{n}\right)+R_i\left(\frac{n-2}{n}\right), & & & \cdot\ , & R_i\left(\frac{n-1}{n}+\frac{\Omega_j}{\Omega_i}\right) \\ Q_i(1), & Q_i\left(\frac{n-1}{n}\right)+R_i\left(\frac{n-1}{n}\right), & & & \cdot\ , & R_i(0) \end{bmatrix} \tag{93}$$

and $D_i =$

$$\begin{bmatrix} Q_j(1), & Q_j\left(\frac{n-1}{n}\right)+R_j\left(\frac{n-1}{n}\right), & & Q_j\left(\frac{1}{n}\right)+R_j\left(\frac{1}{n}\right), & R_j(0) \\ Q_j\left(\frac{\Omega_i}{n\Omega_j}+1\right), & Q_j\left(\frac{\Omega_i}{n\Omega_j}+\frac{n-1}{n}\right)+R_j\left(\frac{\Omega_i}{n\Omega_j}+\frac{n-1}{n}\right), & & Q_j\left(\frac{\Omega_i}{n\Omega_j}+\frac{1}{n}\right)+R_j\left(\frac{\Omega_i}{n\Omega_j}+\frac{1}{n}\right), & R_j\left(\frac{\Omega_i}{n\Omega_j}\right) \\ & & Q_j\left(\frac{2\Omega_i}{n\Omega_j}+\frac{n-2}{n}\right)+R_j\left(\frac{2\Omega_i}{n\Omega_j}+\frac{n-2}{n}\right), \cdot\ , & & \\ Q_j\left(\frac{n-1}{n}\frac{\Omega_i}{\Omega_j}+1\right), & Q_j\left(\frac{n-1}{n}\frac{\Omega_i}{\Omega_j}+\frac{n-1}{n}\right)+R_j\left(\frac{n-1}{n}\frac{\Omega_i}{\Omega_j}+\frac{n-1}{n}\right), & & & \\ Q_j\left(1+\frac{\Omega_i}{\Omega_j}\right), & Q_j\left(\frac{\Omega_i}{\Omega_j}+\frac{n-1}{n}\right)+R_j\left(\frac{\Omega_i}{\Omega_j}+\frac{n-1}{n}\right), & & \cdot\ , & R_j\ \frac{\Omega_i}{\Omega_j} \end{bmatrix} \tag{94}$$

Where $j = 2$ when $i = 1$ and $j = 1$ when $i = 2$,

$$Q_i(u) = \frac{n}{\Omega_i^2} \sum_{s=1}^{\infty} \frac{b_s}{\beta_s^2} \frac{\exp(-\Omega_j \beta_s u)}{1 - \exp[-\beta_s(\Omega_1 + \Omega_2)]} \left[\exp\left(\frac{\beta_s \Omega_i}{n}\right) - 1 - \frac{\beta_s \Omega_i}{n}\right] \tag{95}$$

and

$$R_i(u) = \frac{n}{\Omega_i^2} \sum_{s=1}^{\infty} \frac{b_s}{\beta_s^2} \frac{\exp(-\Omega_j \beta_s u)}{1 - \exp[-\beta_s(\Omega_1 + \Omega_2)]} \left[\exp\left(-\frac{\Omega_i \beta_s}{n}\right) - 1 + \frac{\Omega_i \beta_s}{n}\right]. \tag{96}$$

To confirm the validity of these generalised forms, example 1 of Table 1 was repeated using equations (89)–(94) and then integrating (87) and (88) along the regenerator by the method already described for integration of equations (31) and (32).

The functions, Q and R were not evaluated as accurately as the functions H. In particular, $R(0)$ may have been in error by as much as 5 per cent (all values of H being computed to within 0·1 per cent). However, from the results shown in Table 2, it can be seen that the heat transfer rate agrees with that of the more accurate computation to better than 3 per cent. This agreement is considered to be sufficiently close to confirm the validity of the generalised algebra.

RESULTS

Several examples have been computed using the method defined by equations (53)–(60), all elements of the matrices A_1' and A_2' being computed to within 0·1 per cent. The step sizes for numerical integration were defined by $m = 3$ and $n = 5$, and this contained the computation time within reasonable bounds. Table 1 shows details of the print-out from one example, together with explanatory comments. An immediate problem is to determine the range of parameters over which such a computation produces reasonably accurate results. A major difficulty is that, at present, there is no

Table 1. Example of print-out from Pegasus digital computer

Example No. 1
Input Data

Ω	M_1	N_1	M_2	N_2
0·4176	0·514943	1·363459	0·33658	0·232786

Print-out	Explanation		
+0·396299			
+0·279602	Values of K_1 to K_7		
+0·239483			
+0·213133	$\sum_{s=1}^{7} K_s = 1$		
+0·192410			
+0·174237	See equation (69)		
−0·495163			
		τ	ζ
−1·000000		0	
−1·000000		0·08352	
−1·000000	f_2	0·16704	
−1·000000		0·25056	
−1·000000		0·33408	
−1·000000		0·41760	
+0·098865		0	−1
+0·215561		0·08352	
+0·255681		0·16704	
+0·282030	f_1	0·25056	
+0·302754		0·33408	
+0·320927		0·41760	
−0·745189			
−0·770942			
−0·779417			
−0·784915			
−0·789210			
−0·792965			$-\frac{2}{3}$
+0·252574			
+0·362915			
+0·400361			
+0·424723			
+0·443744			
+0·460330			

Table 1. *contd.*

	ζ
−0·513010	
−0·557367	
−0·572116	
−0·581726	
−0·589259	
−0·595858	$-\frac{1}{3}$
+0·397923	
+0·502050	
+0·536493	
+0·558571	
+0·575621	
+0·590371	
−0·300690	
−0·358084	
−0·377363	
−0·389980	
−0·399900	
−0·408612	0
+0·536864	
+0·634043	
+0·664617	
+0·683781	
+0·698356	
+0·710831	
−0·105994	
−0·172053	
−0·194475	
−0·209215	
−0·220843	
−0·231079	$+\frac{1}{3}$
+0·673720	
+0·760250	
+0·785055	
+0·800134	
+0·811373	
+0·820861	
+0·072819	
+0·001606	
−0·022841	
−0·038993	
−0·051782	
−0·063068	$+\frac{2}{3}$
+0·819091	
+0·882125	
+0·897504	
+0·906523	
+0·913080	
+0·918528	
+0·236977	
+0·163591	
+0·138052	
+0·121072	
+0·107567	
+0·095613	+1
+0·999999	
+1·000001	
+1·000000	
+1·000000	
+1·000000	
+1·000000	

Table 1. *contd.*

+0·336580	M_2	
+0·514943	M_1	
+1·137542	Mean change in $f_2 = \overline{\delta f_2}$*	
+0·743975	Mean change in $f_1 = \overline{\delta f_1}$*	
+0·382874	$M_2 \times \overline{\delta f_2}$	Equality of these quantitities
+0·383105	$M_1 \times \overline{\delta f_1}$	confirms heat balance

* Computed using the Newton–Cotes type of approximation. In this case the heat balance error is (0·383105 − 0·382874)/0·382874 × 100% = 0·06%.
Hausen's method [6] gives $M_1\overline{\delta f_1} = M_2\overline{\delta f_2} = 0{\cdot}3871$.

method available to determine the error of computation absolutely. A heat balance test at the end of the computation is a relevant factor, but this cannot be numerically related to the error in the computed rate of heat transfer. Initially, the validity of the computation was confirmed [8] by comparing the computed mean preheat level with that calculated using Hausen's method [6]. For most examples the heat balance error was less than 1 per cent, and in these cases agreement with the Hausen method was within the limits imposed by reading values from Hausen's graphical presentation of various parameters. Among these successfully computed examples [8] were two for which previous workers obtained badly balanced results when attemptint to use the Collins and Daws' method. The authors currently regard a computation with a heat balance error of less than 1 per cent as being reasonably accurate.

From the computed examples, two effects are worthy of note.

1. The computed heat balance error increases with increasing values of Ω.
2. For isolated examples, even with low values of Ω, the computation has produced results which are obviously absurd.

The effect of increasing Ω is shown in Table 3. In general, Ω is usually less than unity for open hearth furnace regenerators, and in this region the computed heat balance is generally less than 1 per cent in error (except for cases of extreme inequality of M_1 and M_2). However,

Table 2. Comparison of gas exit temperatures computed by original and generalised algebra
Data given: As Example 1, Table 1.

ζ	τ	Original	Generalised
		f_1	
-1	0	$+0{\cdot}098865$	$+0{\cdot}166138$
	0·08352	$+0{\cdot}215561$	$+0{\cdot}239495$
	0·16704	$+0{\cdot}255681$	$+0{\cdot}268053$
	0·25056	$+0{\cdot}282030$	$+0{\cdot}296076$
	0·33408	$+0{\cdot}302754$	$+0{\cdot}314295$
	0·41760	$+0{\cdot}320927$	$+0{\cdot}330435$
		f_2	
$+1$	0	$+0{\cdot}236977$	$+0{\cdot}209140$
	0·08352	$+0{\cdot}163591$	$+0{\cdot}139073$
	0·16704	$+0{\cdot}138052$	$+0{\cdot}114244$
	0·25056	$+0{\cdot}121072$	$+0{\cdot}102454$
	0·33408	$+0{\cdot}107567$	$+0{\cdot}089450$
	0·41760	$+0{\cdot}095613$	$+0{\cdot}077950$
Heat balance			
Mean change in $f_2 = \overline{\delta f_2}$		$+1{\cdot}137542$	$+1{\cdot}116072$
Mean change in $f_1 = \overline{\delta f_1}$		$+0{\cdot}743975$	$+0{\cdot}725085$
$M_2 \times \overline{\delta f_2}$		$+0{\cdot}382874$	$+0{\cdot}375893$
$M_1 \times \overline{\delta f_1}$		$+0{\cdot}383105$	$+0{\cdot}373377$
% error in heat balance		0·06	0·67

for blast furnace stoves, Ω may be of the order of 6·0. In this region more accurate finite difference forms are needed, or the intervals for stepwise integration should be reduced.

In isolated examples where absurd results have been obtained, it has always been possible to obtain a sensible computation for the "inverted" example (obtained by inter-changing the values of M_1 and N_1 with those of M_2 and N_2). From symmetry, the solution of any problem, can be obtained from the solution of its invert. Table 4 illustrates one example of this type. Since the inverted problems are computed accurately and sensibly, it is concluded that the difficulty is probably due to singularities arising in the matrix arithmetic, rather than instability of the finite difference forms.

Table 4. Example of computation producing absurd results

Example No.	Ω	M_1	N_1	M_2	N_2
7	0·015	0·1	0·6	0·5	0·45
8	0·015	0·5	0·45	0·1	0·6

Computation of example 7 produced values of f_1 and f_2 ranging from -10^3 to $+10^8$, whereas heat transfer laws limit the possible range from $-1{\cdot}0$ to $+1{\cdot}0$. By inverting example 7 (i.e. interchanging M_1 and N_1 with M_2 and N_2) to become example 8, no anomalies occurred in the computation, and the following result was obtained

$M_1\overline{\delta f_1}$	$M_2\overline{\delta f_2}$	Heat balance error, %	Hausen's Method $M_1\overline{\delta f_1} = M_2\overline{\delta f_2}$
0·197441	0·197583	0·07	0·193

From the symmetry of the problem, $M_i\overline{\delta f_i}$ must be the same for the original example 7 as for its invert, example 8.

In Table 2, the results for Example 1 (see Table 1), obtained using the generalised algebra [equations (87)–(96)], are compared with the results obtained using equations (53)–(60). The functions Q and R were summed to only 20 terms in order to avoid overflow in the course of computation (the programme for computing A_1' and A_2' included a facility for accurately estimating the residuals of slowly convergent

Table 3. Effect of Ω on preheat level and on accuracy of computation
In examples 2–6, M_1, N_1, M_2 and N_2 were constant at their respective values as in example 1 (see Table 1).

Example no.	Ω	Computed results $M_1\overline{\delta f_1}$	$M_2\overline{\delta f_2}$	Discrepancy in heat balance, %	Hausen's method $M_1\overline{\delta f_1} = M_2\overline{\delta f_2}$
2	0·1	0·391220	0·391299	0·02	0·390
3	0·5	0·382103	0·381772	0·087	0·380
4	1·0	0·379383	0·378376	0·266	0·376
5	3·0	0·373714	0·368743	1·35	0·365
6	6·0	0·353491	0·342176	3·31	0·343

series). Neglect of all terms after the 20th in the summation of the series Q and R could have produced an error as large as 5·25 per cent for the numerical value of $R(0)$. However, $R(0)$ converges particularly slowly, and the summation of all other Q and R probably approached the preselected accuracy of 0·1 per cent, to which all individual terms of these series were calculated. Since all terms of A'_1 and A'_2 were computed to within 0·1 per cent, the results obtained using these matrices are probably more accurate than those obtained using the more generalised algebra. However, it can be seen from Table 2 that the heat transfer rates computed by these two methods agree to within 3 per cent, which was considered to be sufficient confirmation of the validity of the generalised algebra.

For computation using equations (53)–(60), an Autocode programme on the Ferranti Pegasus digital computer required approximately 16 min to compute elements of A'_1 and A'_2 to an accuracy of $\pm 0{\cdot}1$ per cent, with $n = 5$. Integration along the regenerator required a further 9 min. For a single example, therefore, the total computation time is approximately 25 min. However, if several examples are computed for a single value of Ω, the average computation time converges to 9 min, since A'_1 and A'_2 need not be recalculated if Ω is invariant. Further development will probably reduce the computation time, since emphasis to date has been on verifying the validity and feasibility of the method. Also by current standards, Pegasus is not a particularly fast computer.

CONCLUSIONS

1. The method of computation outlined by Collins and Daws is valid, and its successful application depends on accurate summation of the infinite series involved, and on the selection of suitable finite difference forms for numerical integration.

2. The method has been adapted to cope with systems where the heating and cooling periods are unequal.

3. Future work will include

(a) a more comprehensive investigation of the factors affecting the accuracy of computation;
(b) an examination of the few isolated examples for which computation produces obviously absurd results; and
(c) development of techniques for more accurate summations of the infinite series Q and R arising in the generalised algebra.

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APPENDIX

Derivation of integro-differential Equations

Note on symbols

In this appendix, the symbols A, B, F, f, f_1, f_2, L, L^{-1}, p and z' are used differently from the definitions in the main list of symbols.

$A(\zeta, p)$ and $B(\zeta, p)$ are arbitrary functions of integration.

F, f, f_1, f_2, are clarified in the text.

For any function, $\mathscr{F}(z)$, of z, the Laplace transform is given by

$$L[\mathscr{F}(z)] = \int_{z=0}^{z=\infty} \mathscr{F}(z) . \exp(-pz) \, dz,$$

and if $\mathscr{F}(z)$ is the Laplace transform of $\mathscr{F}_1(z)$ then

$$L^{-1}[\mathscr{F}(z)] = \mathscr{F}_1(z).$$

z' is used as a dummy variable.

Introduction

The method follows closely that given by Collins and Daws. However, their work has not appeared in generally available literature and their derivation was stated for the special case of equal heating and cooling periods [equation (18)]. For these reasons the derivation of equation (71) is outlined here in some detail. When heating and cooling periods are of equal duration, equation (71) reduces to equation (18).

Derivation

Taking the Laplacian Transform of equation (11), we obtain

$$\frac{\partial^2 \bar{\Xi}}{\partial \xi^2} - p\bar{\Xi} = -\Xi(\xi, \zeta, 0) \tag{A.1}$$

where $\bar{\Xi}(\xi, \zeta, p)$ is the Laplace transform of $\Xi(\xi, \zeta, z)$.

The solution of (A.1) is

$$\bar{\Xi} = A(\zeta, p) \cosh(\xi - 1)\sqrt{p} + B(\zeta, p) \sinh(\xi - 1)\sqrt{p} - \int_0^{\xi} \Xi(\xi_1, \zeta, 0) \sinh(\xi - \xi_1)\sqrt{p} \, d\xi_1. \tag{A.2}$$

Equation (12) requires $(\partial\bar{\Xi}/\partial\xi) = 0$ when $\xi = 1$. By applying this boundary condition after differentiating (A.2), we obtain an expression for $B(\zeta, p)$. This is now substituted into (A.2), and the resulting equation is applied at $\xi = 0$, giving

$$\bar{\Xi}(0, \zeta, p) = A(\zeta, p) \cosh\sqrt{p} - \frac{\sinh\sqrt{p}}{\sqrt{p}} \int_0^1 \Xi(\xi, \zeta, 0) \cosh(1 - \xi)\sqrt{p} \, d\xi. \tag{A.3}$$

$A(\zeta, p)$ is now eliminated between (A.3) and its differential, giving the result

$$\bar{\Xi}(0, \zeta, p) = \frac{1}{f(p)} \int_0^1 \Xi(\xi, \zeta, 0) \cosh(1 - \xi)\sqrt{p} \, d\xi - \frac{\cosh\sqrt{p}}{f(p)} \left(\frac{\partial\bar{\Xi}}{\partial\xi} - \bar{\Xi}\right)_{\xi=0} \tag{A.4}$$

where

$$f(p) = \cosh\sqrt{p} + \sqrt{p}\sinh\sqrt{p}. \tag{A.5}$$

To invert (A.4), we first find the inverse of $F(p) = \dfrac{\cosh\sqrt{p}}{\sqrt{p}}$.

Using $L^{-1}\{F(z)\} = \dfrac{1}{2\pi i} \displaystyle\int_{\gamma - i\infty}^{\gamma + i\infty} \exp(pz) F(p) \, dp$

$=$ sum of residues of $F(p)$ at all its poles, and all poles are on one side of γ [15],

we have

$$\frac{\cosh\sqrt{p}}{f(p)} = L\left(\sum_{s=1}^{\infty} b_s \exp(-\beta_s z)\right).$$

Thus (A.4) becomes:

$$\bar{\Xi}(0, \zeta, p) + L\left(\sum_{s=1}^{\infty} b_s \exp(-\beta_s z)\right) L\left(\frac{\partial\Xi}{\partial\xi} - \Xi\right)_{\xi=0} = \frac{1}{f(p)} \int_0^1 \Xi(\xi, \zeta, 0) \cosh(1 - \xi)\sqrt{p} \, d\xi \tag{A.6}$$

where $-\beta_s$ are the roots of $\cosh\sqrt{p} + \sqrt{p}\sinh\sqrt{p} = 0$ (the more convenient equivalent is $\sqrt{\beta_s}\tan\sqrt{\beta_s} = 1$).

Similarly

$$\frac{1}{f(p)} \int_0^1 \Xi(\xi, \zeta, 0) \cosh(1 - \xi)\sqrt{p} \, d\xi = L\left[\sum_{s=1}^{\infty} b_s \exp(-\beta_s z) \int_0^1 \Xi(\xi, \zeta, 0) \frac{\cos(1 - \xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} d\xi\right]. \tag{A.7}$$

Putting this into (A.6) and using

$$L[f_1(z)] . L[f_2(z)] = L\{\int_0^z f_1(z') f_2(z - z') \, dz'\}$$

we obtain on inversion

$$\Xi(0,\zeta,z) + \int_0^{z}\left[\frac{\partial \Xi(0,\zeta,z')}{\partial \xi} - \Xi(0,\zeta,z')\right] \times \sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(z-z')\right] dz'$$

$$= \sum_{s=1}^{\infty} b_s \exp(-\beta_s z) \int_0^1 \Xi(\xi,\zeta,0) \frac{\cos(1-\xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} d\xi. \quad \text{(A.8)}$$

We now apply (A.8) first at $z = \tau$ and then at $z = \tau + n(\Omega_1 + \Omega_2)$

$$\Xi(0,\zeta,\tau) + \int_0^{\tau}\left[\frac{\partial \Xi(0,\zeta,z)}{d\xi} - \Xi(0,\zeta,z)\right] \times \sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(\tau-z)\right] dz = \sum_{s=1}^{\infty} b_s \exp(-\beta_s \tau) \times \int_0^1 \Xi(\xi,\zeta,0) \frac{\cos(1-\xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} d\xi. \quad \text{(A.9)}$$

$$\Xi[0,\zeta,\tau + n(\Omega_1+\Omega_2)] + \int_0^{\tau+n(\Omega_1+\Omega_2)}\left[\frac{\partial \Xi(0,\zeta,z)}{\partial \xi} - \Xi(0,\zeta,z)\right] \times \sum_{s=1}^{\infty} b_s \exp\{-\beta_s[\tau + n(\Omega_1+\Omega_2) - z]\} dz$$

$$= \sum_{s=1}^{\infty} b_s \exp\{-\beta_s[\tau + n(\Omega_1+\Omega_2)]\} \times \int_0^1 \Xi(\xi,\zeta,0) \frac{\cos(1-\xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} d\xi. \quad \text{(A.10)}$$

For the definite integral on the left hand side of (A.10), we use

$$\int_0^{\tau+n(\Omega_1+\Omega_0)} F(z)\,dz = \int_0^{\Omega_1+\Omega_2} F(z)\,dz + \int_{\Omega_1+\Omega_2}^{2(\Omega_1+\Omega_2)} F(z)\,dz \ldots + \int_{(n-1)(\Omega_1+\Omega_2)}^{n(\Omega_1+\Omega_2)} F(z)\,dz + \int_{n(\Omega_1+\Omega_2)}^{n(\Omega_1+\Omega_2)+\tau} F(z)\,dz$$

and for each interval, omitting the first, we successively make the substitutions

$$\mathcal{T} = z - (\Omega_1 + \Omega_2), \mathcal{T} = z - 2(\Omega_1 + \Omega_2), \ldots \mathcal{T} = z - n(\Omega_1 + \Omega_2).$$

Ξ and $\partial\Xi/\partial\xi$ are unaffected by these substitutions, because Ξ is cyclic, of period $(\Omega_1 + \Omega_2)$. For the same reason, the first term on the left hand side of (A.10) can be written $\Xi(0,\zeta,\tau)$.

In this way we obtain from (A.10)

$$\Xi(0,\zeta,z) + \int_0^{\Omega_1+\Omega_2}\left[\frac{\partial \Xi(0,\zeta,z)}{\partial \xi} - \Xi(0,\zeta,z)\right] \times \sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(\tau-z)\right] \sum_{r=1}^{n} \exp\left[-\beta_s r(\Omega_1+\Omega_2)\right] dz$$

$$+ \int_0^{\tau} \frac{\partial \Xi(0,\zeta,z)}{d\xi} - \Xi(0,\zeta,z) \sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(\tau-z)\right] dz$$

$$= \sum_{s=1}^{\infty} b_s \exp\{-\beta_s[\tau + n(\Omega_1+\Omega_2)]\} \int_0^1 \Xi(\xi,\zeta,0) \frac{\cos(1-\xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} d\xi. \quad \text{(A.11)}$$

Subtracting (A.11) from (A.9), we obtain

$$- \int_0^{\Omega_1+\Omega_2}\left[\frac{\partial \Xi(0,\zeta,z)}{\partial \xi} - \Xi(0,\zeta,z)\right] \sum_{s=1}^{\infty} b_s \exp\left[-\beta_s(\tau-z)\right] \times \exp\left[-\beta_s(\Omega_1+\Omega_2)\right]\left[\frac{\exp\left[-\beta_s(\Omega_1+\Omega_2)\right]}{1-\exp\left[-\beta_s(\Omega_1+\Omega_2)\right]}\right] dz$$

$$= \sum_{s=1}^{\infty} b_s \exp(-\beta_s) \{1 - \exp[-\beta_s n(\Omega_1 + \Omega_2)]\}$$

$$\times \int_0^1 \Xi(\xi, \zeta, 0) \frac{\cos(1 - \xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} \mathrm{d}\xi. \qquad (A.12)$$

(A.12) is valid for all n, and in particular, for large n, it reduces to

$$- \int_0^{\Omega_1 + \Omega_2} \left[\frac{\partial \Xi(0, \zeta, z)}{\partial \xi} - \Xi(0, \zeta, z)\right] \sum_{s=1}^{\infty} b_s \exp[-\beta_s(\tau - z)]$$

$$\times \frac{\exp[-\beta_s(\Omega_1 + \Omega_2)]}{1 - \exp[-\beta_s(\Omega_1 + \Omega_2)]} \mathrm{d}z = \sum b_s \exp(-\beta_s)$$

$$\times \int_0^1 \Xi(\xi, \zeta, 0) \frac{\cos(1 - \xi)\sqrt{\beta_s}}{\cos\sqrt{\beta_s}} \mathrm{d}\xi. \qquad (A.13)$$

Substituting this result into (A.9) we obtain equation (71), which reduces to equation (18) for $\Omega_1 = \Omega_2 = \Omega$.

CALCUL DES TEMPÉRATURES TRANSITOIRES DANS DES RÉGÉNÉRATEURS

Résumé—Une méthode non itérative pour calculer les variations de température dans un régénérateur en équilibre cyclique a été développée par des auteurs précédents. Des essais antérieurs pour appliquer cette technique ont donné des résultats erronés mais les auteurs de cet article ont obtenu des résultats raisonnables pour des exemples particuliers sans diminuer la dimension du pas de l'intégration numérique. Ceci a été complété en s'assurant que certaines séries apparaissant dans le calcul sont sommées correctement et en choisissant des formes suffisamment précises pour l'intégration numérique.

La méthode avait déjà été dérivée pour des régénérateurs opérant avec des périodes égales de chauffage et de refroidissement mais a été ici modifiée pour le cas le plus général. On indique la direction d'un développement ultérieur.

BERECHNUNG DER INSTATIONÄREN TEMPERATURVERTEILUNG IN REGENERATOREN

Zusammenfassung—In Arbeiten anderer Forscher ist eine nicht iterative Methode zur Berechnung der Temperaturänderungen in einem Regenerator im zyklischen Gleichgewicht entwickelt worden. Frühere Versuche, diese Technik anzuwenden, haben ungenaue Ergebnisse geliefert, es ist jedoch den oben genannten Autoren gelungen, für ausgewählte Beispiele vernünftige Resultate zu erzielen, ohne dabei die Schrittweite für die numerische Integration verringern zu müssen,

Dies wurde erreicht, indem die exakte Summierung einiger in der Berechnung auftretender unendlicher Reihen sichergestellt und genügend genaue Formen für die numerische Integration ausgewählt wurden.

Die Methode wurde zuerst für Regeneratoren mit gleichlangen Heiz- und Kühlphasen entwickelt, ist aber jetzt für den allgemeinen Fall modifiziert worden. Die Richtung für weitergehende Untersuchungen wird aufgezeigt.

РАСЧЕТ ПЕРЕХОДНЫХ ТЕМПЕРАТУР В РЕГЕНЕРАТОРАХ

Аннотация—Ранее был развит безитерационный метод расчета изменения температуры в регенераторе с циклическим равновесием. Предыдущие попытки использовать эту методику дали неточные результаты, но авторы настоящей работы получили надежные данные для избранных примеров, не уменьшая размера шага численного интегрирования. Этого удалось добиться, удостоверившись в том, что определенные бесконечные ряды, возникающие при расчетах, суммируются путем выбора точного метода численного интегрирования. Прежде метод был разработан для регенераторов, работающих с равными периодами нагрева и охлаждения, а сейчас модифицирован для более общего случая. Намечено направление дальнейших разработок.